AD-A082 105 COLORADO STATE UNIV FORT COLLINS DEPT OF MATHEMATICS F/G 12/1
ON THE EXISTENCE OF STRONG UNICITY OF ARBITRARILY SMALL ORDER.(U)
FEB 80 B L CHALMERS, G D TAYLOR F49620-79-C-0124
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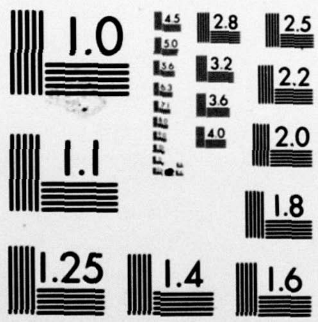

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 19 18 AFOSR-TR-80-0151
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): 6 ON THE EXISTENCE OF STRONG UNICITY OF ARBITRARILY SMALL ORDER.
5. TYPE OF REPORT & PERIOD COVERED: 9 Interim Report
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): 10 B. L. Chalmers [redacted] G. D. Taylor
8. CONTRACT OR GRANT NUMBER(s): 15 F49620-79-C-0124, VNSF-MCS76-08518
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Colorado State University, Department of Mathematics, Fort Collins, CO 80523
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 16 2304 17 A3
11. CONTROLLING OFFICE NAME AND ADDRESS: AFOSR/NM, Bldg. 410, Bollin AFB, D.C. 20332
12. REPORT DATE: 11 Feb[redacted]980
13. NUMBER OF PAGES: 6 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 8
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release, distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):

DTIC ELECTE MAR 18 1980 S D B

18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Uniform approximation theory with constraints, strong uniqueness.

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with the constraints f-P > or = to f-Psubf + delta X P-Psubf to the 1/2m power

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that the strong uniqueness theorem need not hold in its standard form when constraints are imposed on a uniform approximation problem. In particular, it is shown that when approximation with polynomials subject to a monotone constraint $(p^1(x) \geq 0)$ and Hermite-Birkhoff interpolating constraints, a best possible result is the inequality

$||f - p|| \geq ||f - p_f|| + \delta\, ||p - p_f||^{1/2m}$ where p is any approximating polynomial satisfying the constraints and the additional condition that (OVER)

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20. Abstract cont.

norm of P < or = M

P sub f

$\|p\| \leq M$ (M fixed) and P_f is the unique best approximation to f from the given class of approximants.

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ON THE EXISTENCE OF STRONG UNICITY OF ARBITRARILY SMALL ORDER

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The strong unicity theorem, first given by Newman and Shapiro (4), may be described as follows: Given $C[a,b]$ and W an n-dimensional Haar subspace of $C[a,b]$. Let $f \in C[a,b]$ and $p_f \in W$ be the best approximation to f from W. Then there exists a positive constant γ, depending only on f, such that

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\| \tag{1.1}$$

for all $p \in W$ where $\|h\| = \max\{|h(t)|: t \in [a,b]\}$, $h \in C[a,b]$. The extension of this theorem to the setting of monotone approximation has recently been studied by Fletcher and Roulier (3) and Schmidt (5). Specifically, fix an interval $[a,b]$, integers $1 \leq r_0 < \ldots < r_k$, signs $\varepsilon_i = \pm 1$, $i=0,\ldots,k$ and define $K = K(r_0,\ldots,r_k;\varepsilon_0,\ldots,\varepsilon_k)$ by

$$K = \{p \in \Pi_n: \varepsilon_j p^{(r_j)}(x) \geq 0,\ a \leq x \leq b,\ j=0,1,\ldots,k \text{ with } k \leq n\} \tag{1.2}$$

where Π_n denotes the class of all real algebraic polynomials of degree $\leq n$. The study of approximation of $C[a,b]$ by K is called the monotone

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[1]Research supported in part by the National Science Foundation, under grant MCS-76-08518.
[2]Research supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under contract F-49620-79-C-0124 and by the National Science Foundation, under grant MCS-78-05847.

approximation problem. Professor G.G.Lorentz has played a major role in the development of the theory for this problem. See (2) for a brief expository treatment of this problem and an extensive bibliography.

In (3), Fletcher and Roulier constructed an example in $K=\{p\in\Pi_3 : p'(x)\geq 0$ on $[-1,1]\}$ which shows that the best result of form (1.1) that could hold in this setting would be where $\|p-p_f\|$ is replaced by $\|p-p_f\|^2$. Also, some positive results were given that were extended by Schmidt (5). In (5) it is proved that given $f\in C[a,b]$, K as defined in (1.2), $p_f\in K$ the best monotone approximation to f and a positive constant M, there exists $\gamma>0$ depending only on f and M such that

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\|^2 \tag{1.3}$$

for all $p \in K$ satisfying $\|p\| \leq M$.

In (5) one has the following definition: If p_f is the best uniform approximation to $f\in C[a,b]$ from W a subset of $C[a,b]$, we say that p_f is <u>strongly</u> <u>unique</u> <u>of</u> <u>order</u> α $(0<\alpha\leq 1)$ if for each $M>0$ there is a constant $\gamma>0$ such that

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\|^{1/\alpha}$$

for all $p\in W$ satisfying $\|p\| \leq M$. Thus, these two papers taken together show that in monotone approximation strong unicity of order 1/2 holds and this is a best possible result.

In this paper we shall show that by taking an appropriate combination of interpolatory constraints with a monotone constraint one obtains an approximation problem in which strong unicity of order $\frac{1}{2m}$, m a positive integer, holds and that this is also a best possible result.

Thus, fix m a positive integer and define $K\subset \Pi_n$ by

$$K = \{p\in\Pi_n : p^{(1)}(x)\geq 0, a\leq x\leq b \text{ and } p^{(2)}(x_0)=\ldots=p^{(2m-1)}(x_0)=0 \text{ for } x_0\in(a,b) \text{ fixed, } n\geq 2m+1\}. \tag{1.4}$$

Now, by referring to the general theory of (1), one can prove that corresponding to each $f\in C[a,b]$, there exists a unique best approximation,

p_f from K to f. The basic tools of this theory are extreme linear functionals (extremals) of the dual of Π_n corresponding to f and a given $p \in K$. In this particular setting the extremals are as follows. Given $f \in C[a,b]$ and $p \in K$, define for $x \in [a,b]$, e_x^0 on $C[a,b]$ by $e_x^0(g)=g(x)$ for all $g \in C[a,b]$ (point evaluation) and for $x \in [a,b]$, and $1 \leq j \leq 2m$, e_x^j on Π_n by $e_x^j(q)=q^{(j)}(x)$ for all $q \in \Pi_n$. The linear functional e_x^0, $x \in [a,b]$, is said to be an extremal for f and p provided $|e_x^0(f-p)| = \|f-p\|$. The linear functional e_x^1, $x \in [a,b]$ is said to be extremal for f and p provided $e_x^1(p)=0$. Whenever e_x^1 is an extremal for f and p and $x \notin \{a,x_0,b\}$ then an additional extremal called an <u>augmented extremal</u> is also present; namely, the extremal e_x^2 for which $e_x^2(p)=0$ must also hold (since $p^{(1)}(x) \geq 0$). If $e_{x_0}^1$ is an extremal for f and p, then the linear functional $e_{x_0}^{2m}$ is an <u>augmented extremal</u> for f and p with $e_{x_0}^{2m}(p)=0$ holding (since $p^{(1)}(x) \geq 0$). If one starts with an extremal set for f and p (which contains $e_{x_0}^2, \dots, e_{x_0}^{2m-1}$) and adds all possible augmented extremals (as described above) to this set, then one has the augmented set of extremals for f and p corresponding to the original extremal set. Observing that these augmented extremal sets always correspond to Hermite-Birkhoff interpolation problems in which every supported block is even, it is relatively straightforward to prove that the maximal augmented extremal set for f and its best approximation, p_f, from K must have n+2 elements which span the dual of Π_n. Thus, K is generalized Haar and uniqueness of best approximations holds (1). In addition, suppose p_f is the best approximation to f from K. Then there exists $k \leq n+2$ extremals (e.g. (2)), $E=\{e_i\}_{i=1}^k$, none of which are augmented extremals, for which $\underset{\sim}{0}$ belongs to the convex hull of $\{\sigma(e)e\colon e \in E\}$ where $\sigma(e)=\mathrm{sgn}(f(x)-p_f(x))$ if $e=e_x^0$ for some $x \in [a,b]$, $\sigma(e)=1$ if $e=e_y^1$ for some $y \in [a,b]$ and $\sigma(e_{x_0}^j)=1$, $j=2,\dots,2m-1$. Then, by adjoining to E the set $E^a=\{$all augmented extremals corresponding to elements of E$\}$ we must have that the set $E^{aug}=E \cup E^a$ contains at least n+2 elements of Π_n^* which will necessarily span Π_n^* by the fact that every supported block in the corresponding Hermite-Birkhoff problem is even. Likewise, we must have that there exists $e \in E$ for which $e = e_x^0$ some $x \in [a,b]$ as otherwise E is also an extremal set for f and $p_f + c$, c any constant, for which $\underset{\sim}{0}$ is in the convex hull of $\{\sigma(e)e: e \in E\}$ violating uniqueness of best approximation. Using these observations we can now prove

THEOREM. Let $f \in C[a,b]$ and $p_f \in K$ be the best approximation to f from K. Given $M > 0$ there exists $\gamma = \gamma(f,M) > 0$ such that for $p \in K$ satisfying $\|p\| \leq M$,

$$\|f - p\| \geq \|f - p_f\| + \gamma\|p - p_f\|^{2m}$$

(i.e. strong unicity of order $\frac{1}{2m}$) and this inequality is best possible.

Proof: The proof is an extension of the techniques of Fletcher and Roulier and Schmidt. If $f \in K$ then $\gamma = (2M)^{1-2m}$ suffices. Thus, assume $f \notin K$. Let $E = \{e_i\}_{i=1}^k$ be a set of k extremals, which contains $\{e_{x_0}^j\}_{j=2}^{2m-1}$ but contains no augmented extremals, for which $\underset{\sim}{0}$ is in the convex hull of $\{\sigma(e)e: e \in E\}$. Set $E^{aug} = E \cup E^a$. Further, define E^0, $E^1 \subset E$, where $e \in E$ is in E^0 if $e = e_x^0$ for some $x \in [a,b]$ and $e \in E^1$ if $e = e_y^1$ for some $y \in [a,b]$. Define the semi-norm $\|\cdot\|'$ on Π_n by $\|q\|' = \max\{|e(q)|: e \in E\}$. Set $Q = \{q = \frac{p_f - p}{\|p_f - p\|'} : \|p_f - p\|' \neq 0 \text{ and } p \in K\}$. We claim that $\inf_{q \in Q} \max_{e \in E^0} \sigma(e)e(q) = \tau > 0$. Indeed, if there exist $q \in Q$ with $\max_{e \in E^0} \sigma(e)e(q) \leq 0$. Then from $q = \frac{p_f - p}{\|p_f - p\|'}$ with $\|p_f - p\|' \neq 0$ and $p \in K$ we see that $e(q) \neq 0$ for some $e \in E$ and $e(q) \leq 0$ for all $e \in E^1$. Thus, $\sigma(e)e(q) \leq 0$ for all $e \in E$ with strict inequality holding at least once. This violates the fact that $\underset{\sim}{0}$ belongs to the convex hull of $\{\sigma(e)e: e \in E\}$. Using this lower bound, we have for $p \in K$ with $\|p_f - p\|' \neq 0$ that there exists $e \in E^0$ for which $\sigma(e)e(p_f - p) \geq \tau\|p_f - p\|'$. Now observe that (as usual)

$$\|f - p\| \leq \|f - p_f\| + \gamma\|p_f - p\|'.$$

As this inequality holds for $\|p_f - p\|' = 0$, we have a strong uniqueness-type result for the seminorm $\|\cdot\|'$. Next, the norm, $\|p\|^* = \max\{|e(p)|: e \in E^{aug}\}$, is introduced. Thus, there exists a constant $\lambda > 0$ such that $\|p\|^* \geq \lambda\|p\| \ \forall\ p \in \Pi_n$. Finally, we claim that there exists $A > 0$ for which $\|p_f - p\|' \geq A(\|p_f - p\|^*)^{2m}$, $\forall\ p \in K$ satisfying $\|p\| \leq M$. First observe that $\|p_f - p\|' = 0$ with $p \in K$ implies

$e(p_f-p)=0\ \forall\ a\ E^{aug}$ so that $\|p_f-p\|^*=0$. Now, for $e\in E$, there exists a constant K_1 for which $|e(p_f-p)|\geq K_1|e(p_f-p)|^{2m}$ as $\|p\|\leq M$. Let $e\in E^{aug}\sim E$ and assume that $e=e^{2m}_{x_0}$ (the augmented extremal corresponding to $e^1_{x_0}$). We claim that there exists $K_2>0$ for which $|e^1_{x_0}(p_f-p)|\geq K_2|e^{2m}_{x_0}(p_f-p)|^{2m}\ \forall\ p\in K$ satisfying $\|p\|\leq M$. If this is not the case, then corresponding to each integer $\nu>0$ there exists $q_\nu\in K$ with $\|q_\nu\|\leq M$ for which $|q'_\nu(x_0)|<\frac{1}{\nu}|q^{(2m)}_\nu(x_0)|^{2m}$. Now we may assume that q_ν converges uniformly to $q\in K$. Clearly, $q'(x_0)=0$. We can write $q'_\nu(x)=q'_\nu(x_0)+\frac{q^{(2m)}_\nu(x_0)}{(2m-1)!}(x-x_0)^{2m-1}+s_\nu(x)(x-x_0)^{2m}=\beta_\nu+\alpha_\nu(x-x_0)^{2m-1}$ $+s_\nu(x)(x-x_0)^{2m}$ where $\beta_\nu\to 0,\alpha_\nu\to 0$ (as $q^{(2m)}(x_0)=0$ since $q\in K$), $|s_\nu(x)|\leq M_1$ for all $x\in[a,b]$, some M_1 independent of ν and $q'_\nu(x)\geq 0\ \forall\ x\in[a,b]$. Thus, $0\leq\beta_\nu+\alpha_\nu(x-x_0)^{2m-1}+M_1(x-x_0)^{2m}$ for $x\in[a,b]$. For ν sufficiently large (so that $x\in(a,b)$), set $x-x_0=-\frac{\alpha_\nu(2m-1)}{2mM_1}$. This gives $\frac{(2m-1)}{M_1}\left(\frac{2mM_1}{2m-1}\right)^{2m}\beta_\nu\leq\alpha_\nu^{2m}$ or that there exists a constant K_1 independent of ν (sufficiently large) such that $|q'_\nu(x_0)|\geq K_1|q^{(2m)}_\nu(x_0)|^{2m}$ which is our desired contradiction. Finally, if $e\in E^{aug}\sim E$ is of the form $e=e^2_y$ some $y\in(a,b)\sim\{x_0\}$, the above argument (modified) shows that there exists K_3 for which $|e^1_y(p_f-p)|\geq K_2|e^2_y(p_f-p)|^2$ $\geq K_3|e^2_y(p_f-p)|^{2m}\ \forall\ p\in K$ satisfying $\|p\|\leq M$ where K_3 is independent of p. By taking A to be the smallest of the constants produced above, we have that $\|p_f-p\|'\geq A(\|p_f-p\|^*)^{2m}$ implying $\|f-p\|\geq\|f-p_f\|+\gamma\|p_f-p\|^{2m}\ \forall\ p\in K$ satisfying $\|p\|\leq M$ with $\gamma=\gamma(M,f)>0$ independent of p.

To show this result is best possible we construct an example. Fix m a positive integer and let r_1,r_2,r_3 denote the three roots of $p_0(x)=x^{2m+1}$ $+2x^{2m}-1$ (note $-2<r_1<-1$, $r_2=-1$, $0<r_3<1$). Define $K=\{p\in\Pi_{2m+1}: p'(x)\geq 0,$ $x\in[r_1,r_3],\ 0=p^{(2)}(0)=\ldots=p^{(2m-1)}(0)\}=\{p(x)=a_0x^{2m+1}+a_1x^{2m}+a_2x+a_3: p'(x)\geq 0$ on $[r_1,r_3]\}$. Define $g\in C[r_1,r_3]$ by $g(r_1)=-\frac{1}{2}$, $g(-1)=\frac{1}{2}$, $g(r_3)=-\frac{1}{2}$ and extend g linearly to all $[r_1,r_3]$. Set $f=g+2x^{2m+1}$ and $p_f(x)=2x^{2m+1}$. Note that $\{-e^0_{r_1},$ $e^0_{-1},-e^0_{r_3},e^1_0\}$ is an extremal set for f and p_f whose convex hull contains the zero of V^*, $V=\{a_0x^{2m+1}+a_1x^{2m}+a_2x+a_3\}$. (Coefficients are: $\alpha_1=1$, $\alpha_2=1+\alpha_3$,

$\alpha_3 = \frac{-r_1^{2m+1}-1}{r_3^{2m+1}+1}$, $\alpha_4 = r_1+\alpha_2+\alpha_3 r_3$, respectively.) Thus, p_f is the desired best approximation to f from $K^{(2)}$. Next, define $p_\alpha(x)=p_f(x)+\alpha p_0(x)+4m\alpha^{2m}x$, for $0<\alpha\leq\alpha_0$ where α_0 is chosen so small that $|f-p_\alpha|=|g-\alpha[p_0+4m\alpha^{2m-1}x]|$ decreases as x moves away from r_i in a neighborhood of $\{r_1,r_2,r_3\}$ for all α $(0<\alpha\leq\alpha_0)$. This can be done since $|g|$ decreases linearly as x moves away from r_i. Hence α_0 can be chosen so small that $\|f-p_\alpha\| = \max_{i=1,2,3} |(f-p_\alpha)(r_i P|$, $0<\alpha\leq\alpha_0$ $=f(-1)-p_\alpha(-1)=\frac{1}{2}+4m\alpha^{2m}$. Also, $\|f-p_f\|=\frac{1}{2}$, $\|p_f-p_\alpha\|\geq|p_f(0)-p_\alpha(0)|=\alpha$ and $p_\alpha'(x)=2(2m+1)x^{2m}+\alpha((2m+1)x^{2m}+4mx^{2m-1})+4m\alpha^{2m}$. Now, for $x>0$, $p_\alpha'(x)>0$; for $x\in[r_1,-\alpha]$, the term $2(2m+1)x^{2m}$ dominates showing that $p_\alpha'(x)>0$ here; and for $x\in[-\alpha,0]$ the term $4m\alpha^{2m}\geq|4\alpha m x^{2m-1}|$ again implying that $p_\alpha'(x)\geq 0$. Thus $p_\alpha\in K$ and $(\|f-p_\alpha\| - \|f-p_f\|)/\|p_f-p_\alpha\|^\beta \leq \frac{4m\alpha^{2m}}{\alpha^\beta}$. This implies that we must have $\beta\geq 2m$ in order for the strong unicity theorem to hold for this f and p_f. ∎

By suitably selecting g, it can be shown that this weaker strong uniqueness result holds for an f which also satisfies all the constraints of K. Additional results on this topic will appear elsewhere.

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